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# On 1–1 Bivariate Transformations

J. MICHAEL FITZPATRICK

*Department of Computer Science, Vanderbilt University, Nashville, Tennessee 37235, U.S.A.*

AND

LARRY L. SCHUMAKER*

*Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, U.S.A.*

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We study a class of one-to-one tensor mappings defined on a rectangular region of the plane. Such mappings are of interest in image processing, computer vision, biological morphology, cartography, and medical imaging. The mappings are constructed as tensor-products of univariate functions, and the main result is a set of constraints on the parameters of the mapping which assure that it is one-to-one. We illustrate the method by showing how to construct one-to-one mappings using tensor-product polynomials and tensor-product splines. 

## INTRODUCTION

Mappings defined on a rectangular region in the plane are important in several areas of image processing, computer vision, biological morphology, cartography, and medical imaging; see [1, 2, 7, 9, 10, 14, 18–20] and the references therein. For these applications, one needs the mappings to be smooth, computationally simple, and one-to-one. Given these requirements, a natural way to construct such mappings is to use tensor-products of simple univariate functions such as polynomials. The problem then becomes one of giving conditions on the parameters of the mapping which guarantee that it is one-to-one. This problem has been studied for bilinear and biquadratic polynomials in [9] and [8], respectively.

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In this paper we consider general tensor-products. Our approach to the problem is simpler and more direct than that in [8, 9], and avoids the analysis of Jacobian matrices. The paper is organized as follows. In Section 2 we establish the main result of the paper concerning 1–1 mappings, and in Section 3 we apply it to tensor-product mappings. In Section 4 we specialize to tensor-product polynomials. In addition to results for polynomials of general degree, we also show that in the case of bilinear tensor-polynomials we recover the result of [9], while for biquadratic tensor-polynomials the much simpler method presented here produces conditions which are only slightly more restrictive than those obtained in [8]. Section 5 is devoted to splines. Our last section contains several remarks.

## 2. The Perturbed Identity Map

Let $H = I \times J$ be a rectangle in the plane, where without loss of generality, we may assume that $I = [-\alpha, \alpha]$ and $J = [-\beta, \beta]$. In this paper we shall be interested in transformations $T\colon H \to \mathbb{R}^2$ defined by

$$T(x, y) = (u(x, y), v(x, y)) \tag{2.1}$$

with

$$u(x, y) = x + a(x, y) \tag{2.2}$$

$$v(x, y) = y + b(x, y), \tag{2.3}$$

where $a(x, y)$ and $b(x, y)$ are continuous functions defined on $H$.

Clearly, we can think of $u(x, y)$ as a perturbation of $x$ by $a(x, y)$, and $v(x, y)$ as a perturbation of $y$ by $b(x, y)$. Thus $T$ is a perturbation of the identity map. If both $a$ and $b$ are zero, then $T$ is the identity mapping, and $H$ is mapped one-to-one onto itself. In general, the set $TH$ will differ from $H$, although it is always a closed connected subset of $\mathbb{R}^2$ with continuous boundary curves, and if the perturbations are small, is close to $H$ (e.g., in the Hausdorff metric).

We can now establish the main result of the paper. We denote the uniform norm of a function $f$ by $\|f\|$.

THEOREM 2.1. *Let $r > 0$ be given. Suppose that $a$ and $b$ are functions on $H$ with partial derivatives satisfying*

$$\|a_x\| < \frac{1}{2}, \qquad \|a_y\| < \frac{r}{2}, \qquad \|b_x\| < \frac{1}{2r}, \qquad \|b_y\| < \frac{1}{2}. \tag{2.4}$$

*Then $T$ is one-to-one on $H$.*

*Proof.* We have to show that if $P_1 := (x_1, y_1)$ and $P_2 := (x_2, y_2)$ are two distinct points in $H$, then $TP_1 \neq TP_2$; i.e.,

$$(u(x_2, y_2), v(x_2, y_2)) \neq (u(x_1, y_1), v(x_1, y_1)).$$

Let $h_x := x_2 - x_1$ and $h_y := y_2 - y_1$. Then

$$\begin{aligned} u(x_2, y_2) - u(x_1, y_1) &= u(x_2, y_2) - u(x_2, y_1) + u(x_2, y_1) - u(x_1, y_1) \\ &= \int_{y_1}^{y_2} u_y(x_2, t)\, dt + \int_{x_1}^{x_2} u_x(s, y_1)\, ds \\ &= \int_{y_1}^{y_2} a_y(x_2, t)\, dt + \int_{x_1}^{x_2} [1 + a_x(s, y_1)]\, ds. \end{aligned}$$

Similarly,

$$v(x_2, y_2) - v(x_1, y_1) = \int_{x_1}^{x_2} b_x(s, y_1)\, ds + \int_{y_1}^{y_2} [1 + b_y(x_2, t)]\, dt.$$

Now there are two cases:

*Case* 1 $(rh_y \leqslant h_x)$. In this case we have

$$u(x_2, y_2) - u(x_1, y_1) \geqslant h_x - \|a_x\|\, h_x - \|a_y\|\, h_y > h_x[1 - \tfrac{1}{2} - \tfrac{1}{2}] = 0,$$

which asserts that $TP_2 \neq TP_1$.

*Case* 2 $(h_x < rh_y)$. In this case we have

$$v(x_2, y_2) - v(x_1, y_1) \geqslant h_y - \|b_x\|\, h_x - \|b_y\|\, h_y > h_y[1 - \tfrac{1}{2} - \tfrac{1}{2}] = 0,$$

which again asserts that $TP_2 \neq TP_1$. ∎

The parameter $r$ in Theorem 2.1 allows some flexibility in satisfying condition (2.4), and indeed, we shall choose different values of $r$ at different times in the applications below.

## 3. Tensor-Product Mappings

Since we are working on a rectangle $H$, it is natural to choose the perturbation functions $a$ and $b$ in (2.2) and (2.3) to be tensor product functions of the form

$$a(x, y) = \sum_{i=1}^{n_a} \sum_{j=1}^{\tilde{n}_a} a_{ij} l_{i,a}(x)\, \tilde{l}_{j,a}(y) \tag{3.1}$$

$$b(x, y) = \sum_{i=1}^{n_b} \sum_{j=1}^{\tilde{n}_b} b_{ij} l_{i,b}(x)\, \tilde{l}_{j,b}(y), \tag{3.2}$$

where $l_{1,a}(x), \ldots, l_{n_a,a}(x)$ and $l_{1,b}(x), \ldots, l_{n_b,b}(x)$ are linearly independent sets of functions defined on $I$, and $\tilde{l}_{1,a}(y), \ldots, \tilde{l}_{\tilde{n}_a,a}(y)$ and $\tilde{l}_{1,b}(y), \ldots, \tilde{l}_{\tilde{n}_b,b}(y)$ are similar sets defined on $J$.

The coefficient matrices $A$ and $B$ can be thought of as *control parameters*. The smoothness of the mapping $T$ depends on the smoothness of the functions appearing in (3.1) and (3.2). For the remainder of this paper we assume at least that they are continuous and have partial derivatives which are integrable.

To use Theorem 2.1 in practice, we need to have a convenient way to check (2.4). The following lemma gives bounds on the partial derivatives of $a$ and $b$ in terms of the matrices $A$ and $B$.

LEMMA 3.1. *Let a be defined as in* (3.1). *Then*

$$\|a_x\| \leqslant \Lambda'_a \tilde{\Lambda}_a \|A\| \qquad \textit{and} \qquad \|a_y\| \leqslant \Lambda_a \tilde{\Lambda}'_a \|A\|, \tag{3.3}$$

*where*

$$\|A\| = \max_{ij} |a_{ij}|,$$

$$\Lambda_a = \max_{x \in I} \sum_{i=1}^{n_a} |l_{i,a}(x)|, \qquad \tilde{\Lambda}_a = \max_{y \in J} \sum_{j=1}^{\tilde{n}_a} |\tilde{l}_{j,a}(y)|, \tag{3.4}$$

$$\Lambda'_a = \max_{x \in I} \sum_{i=1}^{n_a} |l'_{i,a}(x)|, \qquad \tilde{\Lambda}'_a = \max_{y \in J} \sum_{j=1}^{\tilde{n}_a} |\tilde{l}'_{j,a}(y)|. \tag{3.5}$$

*Analogous bounds also hold for* $\|b_x\|$ *and* $\|b_y\|$ *in terms of* $\|B\|$.

*Proof.* Since

$$a_x(x, y) = \sum_{i=1}^{n_a} \sum_{j=1}^{\tilde{n}_a} a_{ij} l'_{i,a}(x) \tilde{l}_{j,a}(y),$$

taking absolute values inside the sums leads immediately to the first inequality in (3.3). The proofs of the other assertions are similar. ∎

We can now combine Theorem 2.1 and Lemma 3.1 to obtain the following

THEOREM 3.2. *Suppose that*

$$\|A\| < \delta_a = \min\left(\frac{1}{2\Lambda'_a \tilde{\Lambda}_a}, \frac{r}{2\Lambda_a \tilde{\Lambda}'_a}\right) \tag{3.6a}$$

*and*

$$\|B\| < \delta_b = \min\left(\frac{1}{2rA'_b\tilde{A}_b}, \frac{1}{2A_b\tilde{A}'_b}\right) \tag{3.6b}$$

*for some* $r > 0$. *Then* $T$ *is one-to-one on* $H$.

In Sections 4 and 5 we apply this theorem using polynomial and spline basis functions. For ease of notation, throughout the remainder of the paper we restrict our attention to the case where $n = n_a = n_b$, and use the same tensor-product basis functions for both of the perturbation functions $a(x, y)$ and $b(x, y)$ defined in (3.1) and (3.2). In this case, dropping the subscripts $a$ and $b$, if we chose $r = A\tilde{A}'/\tilde{A}A'$, then $\delta = 1/(2A'\tilde{A})$ and $\tilde{\delta} = 1/(2A\tilde{A}')$.

## 4. Bivariate Polynomials

In the previous section we made no special assumptions on the functions appearing in (3.1) and (3.2) other than that they should be continuous and have integrable partial derivatives. A natural choice for these functions is to take them to be polynomials. Polynomials have the advantage that they are infinitely differentiable, so that $T$ is a very smooth mapping. In addition, polynomials can be efficiently evaluated by Horner's scheme (or if values on a raster are needed, by even more efficient raster methods, see [17, 21]).

Suppose now that

$$-\alpha = \xi_1 < \xi_2 < \cdots < \xi_n = \alpha \tag{4.1}$$

are points in $I$, and that $l_1(x), ..., l_n(x)$ are the corresponding classical *Lagrange polynomials* of degree $n-1$ given by

$$l_i(x) = \frac{\prod_{k \neq i} (x - \xi_k)}{\prod_{k \neq i} (\xi_i - \xi_k)}, \qquad i = 1, ..., n. \tag{4.2}$$

Similarly, let $\tilde{l}_1(y), ..., \tilde{l}_{\tilde{n}}(y)$ be the Lagrange polynomials of degree $\tilde{n}-1$ associated with the points

$$-\beta = \eta_1 < \eta_2 < \cdots < \eta_{\tilde{n}} = \beta \tag{4.3}$$

in $J$. Then it is clear that the perturbation functions $a$ and $b$ interpolate in the sense that

$$a(\xi_i, \eta_j) = a_{ij}, \qquad 1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant \tilde{n} \tag{4.4}$$

and

$$b(\xi_i, \eta_j) = b_{ij}, \qquad 1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant \tilde{n}. \tag{4.5}$$

This shows that when we use Lagrange polynomials to construct our mapping $T$, the control parameters are nothing more than the values of the perturbations at the grid points. For polynomials, the quantities $\Lambda$ and $\tilde{\Lambda}$ arising in Section 3 are the standard Lebesgue constants about which much is known; see Remark 3.

We now present two simple applications of Theorem 3.2 involving tensor polynomials. Our first theorem uses linear polynomials and recovers a result obtained in [9] by other means.

THEOREM 4.1. *Let $n = \tilde{n} = 2$, and suppose that the grid points are chosen to be $\{-\alpha, \alpha\}$ in $I$ and $\{-\beta, \beta\}$ in $J$. Then a sufficient condition for $T$ to be one-to-one on $H$ is that $\|A\| < \alpha/2$ and $\|B\| < \beta/2$.*

*Proof.* Here the Lagrange functions on $I$ are the linear polynomials

$$l_1(x) = \frac{(\alpha - x)}{2\alpha} \qquad \text{and} \qquad l_2(x) = \frac{(\alpha + x)}{2\alpha}.$$

Those on $J$ are given by

$$l_1(y) = \frac{(\beta - y)}{2\beta} \qquad \text{and} \qquad l_2(y) = \frac{(\beta + y)}{2\beta}.$$

In this case $a$ and $b$ interpolate at the four corners of the rectangle. Clearly we have $\Lambda = \tilde{\Lambda} = 1$, $\Lambda' = 1/\alpha$, and $\tilde{\Lambda}' = 1/\beta$. Now we may apply Theorem 3.2 using $r = \alpha/\beta$. ▮

Our second application involves using quadratic polynomials as discussed in [8].

THEOREM 4.2. *Let $n = \tilde{n} = 3$, and suppose that the grid points are chosen to be $\{-\alpha, 0, \alpha\}$ in $I$ and $\{-\beta, 0, \beta\}$ in $J$. Then a sufficient condition for $T$ to be one-to-one on $H$ is that $\|A\| < \alpha/10$ and $\|B\| < \beta/10$.*

*Proof.* Here the Lagrange functions on $I$ are given by

$$l_1(x) = \frac{x(x-\alpha)}{2\alpha^2}, \qquad l_2(x) = \frac{2(\alpha^2 - x^2)}{2\alpha^2}, \qquad \text{and} \qquad l_3(x) = \frac{x(x+\alpha)}{2\alpha^2},$$

and those on $J$ are similar. In this case the mapping is defined by interpolation at 9 grid points. It is easy to see that

$$\Phi(x) := \sum_{i=1}^{3} |l_i(x)| = \begin{cases} (\alpha^2 - x\alpha - x^2)/\alpha^2, & -\alpha \leqslant x \leqslant 0 \\ (\alpha^2 + x\alpha - x^2)/\alpha^2, & 0 \leqslant x \leqslant \alpha. \end{cases}$$

This function takes a maximum value of 5/4 at $x = \pm\alpha/2$. Thus $\Lambda = \tilde{\Lambda} = 5/4$. To calculate $\Lambda'$, we need to consider the function

$$\Psi(x) := \sum_{i=1}^{3} |l_i'(x)|.$$

This function is made up of linear polynomials joined together smoothly at the three knots $-\alpha/2, 0$, and $\alpha/2$. For example, it is easy to see that for $x$ in the interval $[\alpha/2, \alpha]$, $\Psi(x) = 4x/\alpha^2$, and that the maximum value on this interval is $4/\alpha$ and is taken on at $x = \alpha$. Checking the other intervals (these functions are always symmetric about 0), we find that the maximum of $\Psi$ on $I$ occurs at $\pm\alpha$, and so we have $\Lambda' = 4/\alpha$. Similarly, $\tilde{\Lambda}' = 4/\beta$. The result follows from Theorem 3.2 if we choose $r = \alpha/\beta$. ∎

A sharper version of Theorem 4.2 was proved in [8], where the constant $\frac{1}{8}$ appears instead of $\frac{1}{10}$. The proof in [8], however, is based on a rather complicated argument involving the Jacobian of the mapping $T$. It was shown there that $\frac{1}{8}$ is sharp in the sense that if larger perturbations are allowed, then non one-to-one transformations can be constructed. We now extend Theorems 4.1 and 4.2 to polynomials of arbitrary degree.

THEOREM 4.3. *Let $n = \tilde{n} \geqslant 4$, and suppose the grid points are chosen to be $\{\alpha \cos((n-1-j)\pi/(n-1))\}_{j=0}^{n-1}$ in $I$ and $\{\beta \cos((n-1-j)\pi/(n-1))\}_{j=0}^{n-1}$ in $J$. Then a sufficient condition for $T$ to be one-to-one on $H$ is that*

$$\|A\| < \alpha/(2C(n-1)^2 \ln(n-1)), \qquad \|B\| < \beta/(2C(n-1)^2 \ln(n-1)), \tag{4.6}$$

*where $C = 2/\pi + 1/\ln(3) \approx 1.54686$.*

*Proof.* Combining Eqs. (6) and (45) in Brutman [5] gives $\Lambda = \tilde{\Lambda} \leqslant 1 + (2/\pi)\ln(n-1) \leqslant C\ln(n-1)$ for $n \geqslant 4$. Moreover, by a result of Berman (see [15]), $\Lambda' \leqslant (n-1)^2/\alpha$ and $\tilde{\Lambda}' \leqslant (n-1)^2/\beta$. The result now follows by Theorem 3.2 with $r = \alpha/\beta$. ∎

Analogous results for other choices of interpolation points can be obtained whenever it is possible to estimate both the $\Lambda$ and $\Lambda'$ constants.

We conclude this section with a different kind of result for polynomials of arbitrary degree. We now use the classical *Bernstein polynomials* (cf. [12]) as basis functions in the expressions (3.1), (3.2) for the perturbation functions $a$ and $b$; i.e., we take

$$l_i(x) := B_{i,n}(x) := \frac{(n-1)!\,(\alpha - x)^{i-1}\,(\alpha + x)^{n-i}}{(i-1)!\,(n-i)!\,(2\alpha)^{n-1}}, \qquad i = 1, ..., n, \tag{4.7}$$

$$\tilde{l}_j(y) := \tilde{B}_{j,\tilde{n}}(y) := \frac{(\tilde{n}-1)!\,(\beta - y)^{j-1}\,(\beta + y)^{\tilde{n}-j}}{(j-1)!\,(\tilde{n}-j)!\,(2\beta)^{\tilde{n}-1}}, \qquad j = 1, ..., \tilde{n}. \tag{4.8}$$

One advantage of using these basis functions is that now the coefficient matrices $A$ and $B$ can be thought of as *control polygons* for the surfaces $a$ and $b$ (cf. [6]), allowing a convenient way to modify the mapping $T$ in a controlled way.

It is clear that these basis functions are positive and sum to one, and thus $\Lambda = \tilde{\Lambda} = 1$. Now

$$B'_{1,n}(x) = -(n-1)\, B_{1,n-1}(x)/2\alpha,$$
$$B'_{i,n}(x) = (n-1)[B_{i-1,n-1}(x) - B_{i,n-1}(x)]/2\alpha, \qquad i = 2, ..., n-1,$$
$$B'_{n,n}(x) = (n-1)\, B_{n-1,n-1}(x)/2\alpha.$$

These facts imply that $\Lambda' \leqslant (n-1)/\alpha$ and $\tilde{\Lambda}' \leqslant (\tilde{n}-1)/\beta$.

THEOREM 4.4. *Let $T$ be defined as in* (2.1)–(2.3) *using the Bernstein basis functions in the expressions* (3.1), (3.2) *defining the perturbation functions $a$ and $b$. Then $T$ is one-to-one on $H$ whenever*

$$\|A\| < \frac{\alpha}{2(n-1)} \qquad \textit{and} \qquad \|B\| < \frac{\beta}{2(\tilde{n}-1)}.$$

*Proof.* The result follows immediately from Theorem 3.2 with $r = \alpha/\beta$. ∎

## 5. SPLINES

Another natural way to define the perturbation functions $a$ and $b$ is to use tensor-product polynomial splines. Given $m > 1$, let $\Delta$ be the partition defined by

$$-\alpha = y_1 = \cdots = y_m \qquad \text{and} \qquad \alpha = y_{n+1} = \cdots = y_{n+m}, \tag{5.1}$$

$$-\alpha < y_{m+1} < y_{m+2} < \cdots < y_n < \alpha, \tag{5.2}$$

and let $N_1^m(x), ..., N_n^m(x)$ be the usual normalized B-splines of order $m$ associated with $\Delta$; see [16]. Similarly, given $\tilde{m} > 1$, let $\tilde{\Delta}$ be the partition defined by

$$-\beta = \tilde{y}_1 = \cdots = \tilde{y}_{\tilde{m}} \qquad \text{and} \qquad \beta = \tilde{y}_{\tilde{n}+1} = \cdots = \tilde{y}_{\tilde{n}+\tilde{m}}, \tag{5.3}$$

$$-\beta < \tilde{y}_{\tilde{m}+1} < \tilde{y}_{\tilde{m}+2} < \cdots < \tilde{y}_{\tilde{n}} < \beta, \tag{5.4}$$

and let $\tilde{N}_1^{\tilde{m}}(y), \ldots, \tilde{N}_{\tilde{n}}^{\tilde{m}}(y)$ be the normalized B-splines of order $\tilde{m}$ associated with $\tilde{\Delta}$. We now define

$$a(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{\tilde{n}} a_{ij} N_i^m(x) \tilde{N}_j^{\tilde{m}}(y) \tag{5.5}$$

$$b(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{\tilde{n}} b_{ij} N_i^m(x) \tilde{N}_j^{\tilde{m}}(y). \tag{5.6}$$

We can now give sufficient conditions for the mapping $T$ to be one-to-one on $H$.

THEOREM 5.1. *Let $T$ be the mapping defined in* (2.1)–(2.3) *using $a$ and $b$ as defined in* (5.5), (5.6), *and let*

$$h = \min_{m \leqslant i \leqslant n} (y_{i+1} - y_i) \qquad \text{and} \qquad \tilde{h} = \min_{\tilde{m} \leqslant i \leqslant \tilde{n}} (\tilde{y}_{i+1} - \tilde{y}_i). \tag{5.7}$$

*Then $T$ is one-to-one on $H$ whenever*

$$\|A\| < h/4 \qquad \text{and} \qquad \|B\| < \tilde{h}/4. \tag{5.8}$$

*If we choose equally spaced knots, these conditions become*

$$\|A\| \leqslant \alpha/2(n - m + 1), \qquad \|B\| \leqslant \beta/2(\tilde{n} - \tilde{m} + 1) \tag{5.9}$$

*Proof.* It is well known (cf. [16]) that the normalized B-splines are positive and add to one. This implies that the constants $\Lambda$ and $\tilde{\Lambda}$ defined in Lemma 3.1 are both equal to 1. Now to bound the constant $\Lambda'$ defined in (3.5), we use the fact (cf. Theorem 4.16 of [16]) that

$$D_x N_i^m(x) = (m-1)\left(\frac{N_i^{m-1}(x)}{(y_{i+m-1} - y_i)} - \frac{N_{i+1}^{m-1}(x)}{(y_{i+m} - y_{i+1})}\right).$$

This implies that

$$\sum_{i=1}^{n} |D_x N_i^m(x)| \leqslant \frac{2}{h} \sum_{i=2}^{n} N_i^{m-1}(x) = \frac{2}{h}.$$

We conclude that $\Lambda' = 2/h$. A similar analysis implies $\tilde{\Lambda}' = 2/\tilde{h}$, and the sufficiency of conditions (5.8) follows. Finally, for equally spaced knots, we have $h = 1/(n + 1 - m)$ and $\tilde{h} = 1/(\tilde{n} + 1 - \tilde{m})$. ∎

The bounds on the sizes of $A$ and $B$ in (5.8) are independent of the order $m$ of the splines, and depend only on the knot spacing. It is easy to see that to get the largest degree of freedom in choosing the parameters $A$ and $B$

(i.e., the largest constants on the right-hand sides of the inequalities in (5.8)), we should choose the knots to be equally spaced.

The coefficients $a_{ij}$ and $b_{ij}$ in (5.5) and (5.6) can be thought of as control parameters for the mapping $T$. Because of the local support properties of the B-splines, if we change the value of one of these parameters, say $a_{ij}$, then it affects the mapping only locally; in particular, for $(x, y)$ in the rectangle $H_{ij} := [y_i, y_{i+m}] \times [\tilde{y}_j, \tilde{y}_{j+\tilde{m}}]$ which supports the B-spline $N_i^m \tilde{N}_j^{\tilde{m}}$. Thus, the matrices $A$ and $B$ can be thought of as *control nets*, and used much in the same way as they are used in designing surfaces in Computer Aided Geometric Design (cf. [6]).

In some applications, it may be desirable to control the mapping by specifying its values on a grid of points lying in $H$. Suppose we are given grid points as in (4.1), (4.3), and suppose we are given matrices $Z = (z_{\nu,\mu})$ and $\tilde{Z} = (\tilde{z}_{\nu,\mu})$ of real numbers. Assume now that

$$N_i^m(\xi_i) > 0, \qquad i = 1, ..., n \tag{5.10}$$

$$\tilde{N}_j^{\tilde{m}}(\eta_j) > 0, \qquad j = 1, ..., \tilde{n}. \tag{5.11}$$

Then it is well-known [16] that we can find $A$ by solving the interpolation problem

$$a(\xi_\nu, \eta_\mu) = z_{\nu,\mu}, \qquad \nu = 1, ..., n, \qquad \mu = 1, ..., \tilde{n}, \tag{5.12}$$

and we can find the coefficients $B$ by solving the interpolation problem

$$b(\xi_\nu, \eta_\mu) = \tilde{z}_{\nu,\mu}, \qquad \nu = 1, ..., n, \qquad \mu = 1, ..., \tilde{n}. \tag{5.13}$$

Indeed, the conditions (5.10), (5.11) ensure that the interpolation problems have a unique solution, and they can be solved efficiently by standard tensor-methods (cf. [16]). For example, (5.12) can be written as

$$G^{\mathrm{T}} A \tilde{G} = Z, \tag{5.14}$$

where $G$ is the $n \times n$ matrix with entries

$$G_{ij} = N_i^m(\xi_j), \qquad i, j = 1, ..., n, \tag{5.15}$$

and where $\tilde{G}$ is the analogous $\tilde{n} \times \tilde{n}$ matrix. We now illustrate this method for linear splines.

THEOREM 5.2. *Let $m = \tilde{m} = 2$ so that we are working with bilinear splines, and suppose that the knots are equally spaced in both* (5.2) *and* (5.4). *Suppose the grid points are chosen as*

$$\xi_i = y_{i+1},\ i = 1, ..., n, \qquad \eta_j = \tilde{y}_{j+1},\ j = 1, ..., \tilde{n}.$$

*Then a sufficient condition for $T$ to be one-to-one on $H$ is that*

$$\|A\| < \alpha/2(n-1) \qquad \textit{and} \qquad \|B\| < \beta/2(\tilde{n}-1). \tag{5.16}$$

*In terms of the values $Z$ and $\tilde{Z}$ of the perturbations $a$ and $b$ on the grid points $(\xi_i, \eta_j)$, a sufficient condition is that $Z$ and $\tilde{Z}$ satisfy the same inequalities as $A$ and $B$.*

*Proof.* In this case, the matrices $G$ and $\tilde{G}$ are the identity matrices, and so $A = Z$ and $B = \tilde{Z}$. The result follows from Theorem 5.1. ∎

For $m$, $\tilde{m} > 2$, it is no longer the case that $A = Z$ and $B = \tilde{Z}$. Thus, in order to apply Theorem 5.1, we need to find bounds on $\|A\|$ and $\|B\|$ in terms of $\|Z\|$ and $\|\tilde{Z}\|$, respectively. It follows directly from (5.14) that

$$\|A\| \leqslant \|\tilde{G}^{-1}\| \, \|G^{-1}\| \, \|Z\|, \tag{5.17}$$

with a similar inequality for $\|B\|$. In general, we can give bounds on the norms of $G^{-1}$ and $\tilde{G}^{-1}$ only for special spline interpolation methods.

To illustrate the kinds of results which are possible, let $m$, $\tilde{m} > 1$ be prescribed, and suppose $\Delta$ and $\tilde{\Delta}$ are the partitions defined in (5.1)–(5.4). Define

$$\xi_i = (y_{i+1} + \cdots + y_{i+m-1})/(m-1), \qquad i = 1, ..., n, \tag{5.18}$$

$$\eta_j = (\tilde{y}_{j+1} + \cdots + y_{j+\tilde{m}-1})/(\tilde{m}-1), \qquad j = 1, ..., \tilde{n}. \tag{5.19}$$

Let $G_m$ and $\tilde{G}_{\tilde{m}}$ be the associated collocation matrices; cf. (5.15). Then the interpolation problem (5.12) has a unique solution. It is easy to see that these choices of interpolation nodes satisfy the conditions (5.10), (5.11).

THEOREM 5.3. *Let $m = \tilde{m} = 3$, and let $T$ be defined in* (2.1) *using $a$ and $b$ as defined in* (5.5), (5.6) *with coefficients $A$ and $B$ chosen so that* (5.12), (5.13) *are satisfied on an equally spaced grid. Then $T$ is one-to-one on $H$ provided that*

$$\|Z\| < \frac{\alpha}{18(n-2)} \qquad \textit{and} \qquad \|\tilde{Z}\| < \frac{\beta}{18(\tilde{n}-2)}. \tag{5.20}$$

*Proof.* It is shown in [13] that $\|G_3^{-1}\| \leqslant 3$ and $\|\tilde{G}_3^{-1}\| \leqslant 3$. Thus (5.20) implies (5.9), and the result follows from Theorem 5.1. ∎

THEOREM 5.4. *Let $m = \tilde{m} = 4$, and let $T$ be defined in* (2.1) *using $a$ and $b$ as in* (5.5), (5.6) *with coefficients $A$ and $B$ chosen so that $a$ and $b$ inter-*

*polate values $Z$ and $\tilde{Z}$ on an equally spaced grid $(\xi_i, \eta_j)$ as in (5.12), (5.13). Then $T$ is one-to-one on $H$ provided that*

$$\|Z\| < \frac{\alpha}{1458(n-3)} \qquad \text{and} \qquad \|\tilde{Z}\| < \frac{\beta}{1458(\tilde{n}-3)}. \tag{5.21}$$

*Proof.* It is shown in [3] that $\|G_4^{-1}\| \leqslant 27$ and $\|\tilde{G}_4^{-1}\| \leqslant 27$. Combining this with (5.17) implies (5.9), and the result follows from Theorem 5.1. This result can be improved to allow a larger range of parameter values since the bound of 27 for $\|G_4^{-1}\|$ and $\|\tilde{G}_4^{-1}\|$ is not sharp; it is conjectured that a sharp bound is probably around 4 or 5. ∎

Similar results could be obtained for larger values of $m$ if we had estimates on the norms of the inverses of the corresponding matrices $G_m^{-1}$. Unfortunately, such estimates do not seem to be available.

There are many other spline interpolation operators which could be used to define the perturbations $a$ and $b$. We conclude this section with one more example involving $C^1$ cubic splines which interpolate in the Hermite sense (with zero derivative values).

THEOREM 5.5. *Suppose we are given equally spaced grid points $(\xi_i, \eta_j)$, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant \tilde{n}$. For each $1 \leqslant i \leqslant n$, let $l_i(x)$ be the unique piecewise cubic $C^1$ function which satisfies $l_i(\xi_j) = \delta_{ij}$ and $l_i'(\xi_j) = 0$, for all $i$, $j = 1, ..., n$. Let $\tilde{l}_1(y), ..., \tilde{l}_{\tilde{n}}(y)$ be the analogous piecewise polynomial functions with respect to the $\eta_j$'s. Then the perturbation functions $a$ and $b$ defined in* (3.1), (3.2) *interpolate as in* (4.4), (4.5), *and the associated mapping $T$ is one-to-one on $H$ whenever*

$$\|A\| < h/6 \qquad \text{and} \qquad \|B\| < \tilde{h}/6. \tag{5.22}$$

*Proof.* Clearly these basis functions are positive and form a partition of unity, and hence $\Lambda = \tilde{\Lambda} = 1$. In addition, it is easy to check that $\Lambda' = 3/h$ and $\tilde{\Lambda}' = 3/\tilde{h}$, where $h$ and $\tilde{h}$ are as in (5.7). The assertion now follows from Theorem 3.2. ∎

## 5. REMARKS

*Remark* 1. For other approaches to building one-to-one mappings for these kinds of applications, see [2, 18–20] and references therein.

*Remark* 2. The conditions in Theorem 3.2 are only sufficient conditions, and not necessary ones.

*Remark* 3. For $n = 2, 3$, the points appearing in Theorem 4.3 are the same points used in Theorems 4.1 and 4.2. It is known (cf. [15]) that the general $n$, this choice of points minimizes the size of $\Lambda'$. They do not simultaneously minimize the size of $\Lambda$, and indeed, the question of finding the interpolation points which do remains open (cf. [4, 5, 11]). Thus, finding the largest possible constants in the conditions (4.6) of Theorem 4.3 is most likely a very difficult problem.

*Remark* 4. It is clear from the examples that, in general, as we increase the complexity of our mapping functions by using a larger number of parameters, the range of values of the parameters which assure the mapping is one-to-one becomes smaller and smaller.

*Remark* 5. In the results of Sections 4 and 5, we have illustrated our methods for the case where we use the same form of perturbation for both $a(x, y)$ and $b(x, y)$. The general theory of Sections 2 and 3 allows the use of different functions. Thus, for example, it is possible to use polynomials of different degrees for $a$ and $b$, or even polynomials for one perturbation function, and splines for the other.

*Remark* 6. We have illustrated the method using polynomials and splines, but in some applications it may be useful to use other basis functions.

*Remark* 7. We have discussed only the case where $H$ is a rectangle in $\mathbb{R}^2$. Clearly the same techniques can be applied when $H$ is a parallelopiped in $\mathbb{R}^3$ or even a more general hypercube in $\mathbb{R}^n$.

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